# NONLOCALITY, ENTANGLEMENT, AND RANDOMNESS IN DIFFERENT CONFLICTING INTEREST BAYESIAN GAMES 

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#### Abstract

We analyse different Bayesian games where payoffs of players depend on the types of players involved in a two-player game. The dependence is assumed to commensurate with the CHSH game setting. For this, we consider two different types of each player (Alice and Bob) in the game, thus resulting in four different games clubbed together as one Bayesian game. Considering different combinations of common interest, and conflicting interest coordination and anti-coordination games, we find that quantum strategies are always preferred over classical strategies if the shared resource is a pure non-maximally entangled state. However, when the shared resource is a class of mixed state, then quantum strategies are useful only for a given range of the state parameter. Surprisingly, when all conflicting interest games (Battle of the Sexes game and Chicken game) are merged into the Bayesian game picture, then the best strategy for Alice and Bob is to share a set of non-maximally entangled pure states. We demonstrate that this set not only gives higher payoff than any classical strategy, but also outperforms a maximally entangled pure Bell state, mixed Werner states, and Horodecki states. We further propose the representation of a special class of Bell inequality- tilted Bell inequality, as a common as well as conflicting interest Bayesian game. We thereafter, study the effect of sharing an arbitrary two-qubit pure state and a class of mixed state as quantum resource in those games; thus verifying that non-maximally entangled states with high randomness help attain maximum quantum benefit. Additionally, we propose a general framework of a two-player Bayesian game for d-dimensions Bell-CHSH inequality, with and without the tilt factor.


Keywords: Bayesian games, conflicting interest games, tilted Bell-CHSH inequality
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## 1 Introduction

Based on EPR's argument [1] of reality and localism, Bell designed an inequality [2, 3] to delimit the boundaries between classical and quantum correlations. The Bell inequality is violated by all two-qubit pure entangled states confirming the presence of non-local correlations in an underlying entangled pure state. These correlations are very important to understand the foundational aspects of quantum theory, and its wide applicability in secure information

[^0]processing, and computation $[1,4,5,6,7]$. With the advent of entanglement and nonlocality, the discussions regarding incommensurability of entanglement and quantum nonlocality continued to exist- in fact it is established that entanglement and nonlocality can be considered as distinct resources for quantum information and computation $[8,9,10,11,12]$. Eventually, it was observed that quantum discord which is a measure of nonlocal correlations even in the absence of entanglement, is a necessary resource for computational speed-up [13, 14]. Although a general intuition further suggests direct correspondence between entanglement, nonlocality and randomness in an experiment, it was established that states with arbitrarily less entanglement and nonlocality can produce randomness close to 2 bits suggesting that maximal entanglement or nonlocality do not coincide with maximum randomness [15].

In order to efficiently analyse the benefits of nonlocality in computational tasks, a special class of games known as Bayesian games $[16,17,18]$ serve as the best tool to represent quantum correlations as they contain the required element of incompleteness. These games contain partial information about the other player; since the type of atleast one player in the game is a random variable. The first link between Bayesian games and nonlocality was proposed to demonstrate the relation between the game's payoffs and Cereceda inequalities [19, 20]. Later, Brunner and Linden showed a direct correspondence between the Bell inequality and payoffs of a general two player Bayesian game [21]. They further discussed that nonlocal correlations play a substantial role in generating efficient quantum strategy for players to win and perform better than any classical strategy in Bayesian games.

In general, Bayesian games were studied and analysed with both players having common interests- either they jointly won or jointly lost the game. Howbeit, in real life scenarios, the interests of players may not always coincide, but may differ on several occasions. Conflicting interest games [22] are those in which both the players have different preferences, like in the case of Battle of Sexes game [23]. On these lines, Pappa et al. [24] found that quantum correlations can also be used to win Bayesian games wherein players have conflicting interests. For this, they formulated a combination of CHSH and Battle of Sexes game, and demonstrated that for any classically correlated strategy, the sum of average payoffs of both players can never exceed $\frac{9}{8}$, and a fair classical equilibrium exists at the average payoff of each player being $\frac{9}{16}$. However, when the players share a maximally entangled two-qubit state and rely on the outcomes of various projective measurements, as an advise to choose their quantum strategy, the picture turns out to be in their favour as the sum of average payoff exceeds the classical bound of $\frac{9}{8}$. The fair quantum equilibrium (where sum of average payoff of both players is 1.28) exists at the projective measurement settings which give maximum violation for the Bell-CHSH inequality, i.e., $2 \sqrt{2}[2,3]$. Followed by this, some interesting conflicting interest games have been proposed, in which the payoffs directly depend on Bell-type inequalities [25, 26, 27]. Further, various three player conflicting interest games [28, 29] have also been presented, where the payoffs hold relation to three-qubit Bell-type inequalities [30].

Apart from Bayesian games, there are various two-player games [31, 32, 33, 34] which demonstrate the advantages of quantum players with respect to their classical counterparts. Although most of the games describe the usefulness of maximally entangled Bell states, few also analyse the behaviour of non-maximally entangled states [35, 36], mixed quantum states [37], and the failure of all quantum strategies [38]. Another pertinent question is the relation of nonlocality with the degree of entanglement present in the system $[8,9,10]$. Since the
foundation of Bayesian games lies on the structure of nonlocal correlations, in this article, we study the effect of entanglement in an underlying state being shared by the players in winning quantum games. For this, we first analyse the game proposed by Pappa et al. [24] using general two-qubit pure Bell states as resources- this allows one to analyse the behaviour of non-maximally entangled states towards playing a conflicting interest game.

For a Bayesian game comprising a conflicting interest game and a common interest game, we find that all pure entangled states quantum strategies surpass the classical limit to win the game; and the total payoffs of players increases with the increase in degree of entanglement of the shared resource. Further, we also describe a quantum game by combining two conflicting interest games, i.e., Battle of the Sexes game [23] and Chicken game [39]. Precisely, the players undergo Chicken game when type of both players is Type 1, or Battle of the Sexes game when type of at least one player is Type 0 [21]. Since both the Battle of the Sexes and Chicken game demonstrate conflicting interests of the players involved, the analysis of quantum strategies for these games is substantial in understanding fully conflicting interest CHSH-type Bayesian games. Surprisingly, we find that the players achieve a better payoff by sharing a set of non-maximally entangled pure states instead of the maximally entangled pure state. Interestingly, in both game settings the total payoff of players has a direct correspondence with the maximum expectation value of the Bell-CHSH operator for the underlying state. Clearly, for CHSH inequality less entanglement always corresponds to a violation lesser than the one obtained using the maximally entangled state [40]. Although, in both game settings, the maximally entangled state violates the Bell-CHSH operator maximally, the setting where two conflicting interest games are merged as a Bayesian game lead to the interesting result that a team of players sharing a set of non-maximally entangled two-qubit pure states will win the game against a team of players sharing the maximally entangled two-qubit Bell state. This result will be very useful in formulation of a game where non-maximally entangled states offer more benefit over the maximally entangled state even in the settings of a CHSH inequality. We further analyse two game settings for sharing of mixed states since mixed states are not explored much as far as game theory is concerned. For this, we consider to use Werner [41] and Horodecki states [42]. Unlike pure states, for mixed states, quantum strategies only offer an advantage over a certain range of state parameters. However, quantum strategies may be more useful than classical strategies for the use of mixed states even in the range where mixed states do not violate the Bell-CHSH inequality despite of being entangled.

Moreover, considering that the use of tilted Bell-CHSH operator leads to high randomness when sharing a non-maximally entangled state, we further propose an efficient demonstration of a tilted version of Bell-CHSH inequality [15] as common interest and conflicting interest Bayesian games. The extra term representing tilt in the expression for the tilted Bell-CHSH inequality prompted us to study the inequality under the framework of game theory. Recently, a conflicting interest Bayesian game having correspondence with tilted Bell-CHSH inequality has been put-forth, which highlights the usefulness of two-qubit pure non-maximally entangled states over two-qubit Bell states in terms of social welfare solution [43, 44]. We however demonstrated a model for 5 -parameter common interest and 10-parameter conflicting game by meticulously combining coordination and anti-coordination games to represent tilted BellCHSH inequality. Various instances of these games have then been discussed to analyse tilted Bell-CHSH inequality in the framework of a quantum game for general two-qubit entangled
pure states as well as mixed states. For pure as well as mixed states, we find that the quantum game where conflicting interest games are merged as a Bayesian game results in a much larger set of non-maximally entangled states offering advantage over the maximally entangled state as opposed to the quantum game where common interest games are merged. On the other hand, the use of a mixed Horodecki state, where common interest games are merged, leads to higher quantum social welfare as compared to the use of same mixed state in a quantum game where conflicting interest games are merged. Similar observation is obtained in case of another important class of mixed states [45]. Furthermore, a general framework for representing a ddimension untilted as well as tilted Bell-CHSH inequality [46] has been proposed under the premise of Bayesian game theory for a pure bipartite entangled states.

## 2 Basic terminologies and concepts in game theory

A game is a competitive activity among more than one rational players which consists of a set of rules, conditions of win and loss, and payoffs. The rules define the constraints of moves a player can opt for in the game. Payoffs are quantified rewards each player gets on performing a certain move or on achieving a fixed goal. While playing a game, each player attempts at performing the best move so as to attain maximum reward or payoff. The final outcome or reward of each player depends on the actions of all players or stakeholders involved in the game. Therefore, game theory is a mathematical model of strategic decision making in a game. The theory was developed by John von Neumann, a mathematician and Oskar Morgenstern, an economist with the aim of solving problems related to economics [47]. The discovery of this theory initiated with the realization that the dynamics in economics had a correspondence with game-playing. In general, any situation where the moves of players affect each other's outcomes, thus involving strategic decision making can be modeled mathematically using game theory. For instance, the study of demand and supply of a product in market as a game can assist in evaluating its optimum cost in a competitive market [48]. Likewise, the phenomenon of public choice for voting can be visualized as a game [49]. Another example of the theory lies in evolutionary biology where survival of the fittest being is modeled as survival games [50].

### 2.1 Different types of games

In this sub-section, we summarize terminologies used for different types of games in game theory.
(i) Cooperative and Non-Cooperative Games: Cooperative games are the ones where players negotiate and agree with each other on adopting strategies while playing [51]. The players are in a coalition, and thus these games are studied separately under cooperative game theory. The traditional games however, are non-cooperative in nature. In non-cooperative games, players do not play as a team; they rather individually opt for the strategy that gives them the maximum reward [51]. Prisoners' dilemma [52] is the best example of non-cooperative games.
(ii) Symmetric and Asymmetric Games : If all the strategies adopted by one player in a game is same as all other players; and the reward achieved by players also remains the same even when the same strategy set is performed by interchanging players, then the
game is termed as a symmetric game [53]. Mostly, two-player games such as prisoners' dilemma [52] and chicken's game [39] are symmetric. On the other hand, asymmetric games are those where the players have a different strategy space to opt from, and/or the reward that a player attains on performing a particular strategy may not be the same as the reward that another player gets on performing the same strategy. An instance of asymmetric game is the ultimatum game [54].
(iii) Zero-sum and Non-zero-sum Games : Zero-sum games are a specific case of constant sum games, where the sum of total payoffs of players should be exactly zero [55]. In such cases, the wining condition for a player becomes the losing situation for the other player- chess, tic-tac-toe, and many such games are examples of zero-sum games. Contrary to this, non-zero sum games are the ones where the sum of total payoff of all players is non-zero [55]. In general, cooperative games are examples of non-zero sum games because the coalition (of players) either wins or looses the game collectively.
(iv) Perfect, Imperfect, Complete, and Incomplete Information Games : If every player knows all the actions previously taken by all other players, then the game is a perfect information game. For instance, tic-tac-toe and chess are perfect information games [56]. On the contrary, imperfect information games like poker, are the ones in which the players do not completely know about the prior moves of all players in the game [23]. Simultaneous move games [57] in general, are imperfect information games. Perfect information games are different from complete information games. The players in a complete information game also know about the strategies, payoffs and types of players in the game but they do not necessarily know about all the prior moves of the players. As a contrast, the players do not have all the information in case of incomplete information games. Bayesian games $[16,17,18]$ are examples of incomplete information games, where atleast one player is unaware of the type of other players in the game. Nature is introduced as an additional player which assigns a type to each player depending on the "probability distribution or prior assumption" of available types [58]. This method enables conversion of incomplete information games to imperfect information games.
(v) Common and Conflicting Interest Games : Conflicting interest games [22, 59, 50] are those in which both the players have different preferences, like in the case of Battle of Sexes game [23]. On the other hand, common interest games are the ones where players do not prefer one strategy over the other, but have similar interests in terms of opting for a particular strategy.

Table 1. Prisoners' Dilemma payoff matrix

| Prisoner 1 Prisoner 2 | Cooperate | Defect |
| :---: | :---: | :---: |
| Cooperate | $-1,-1$ | $-3,0$ |
| Defect | $0,-3$ | $-2,-2$ |

### 2.2 Nash equilibrium

For a static game with finite set of strategies for players, John F. Nash [60, 53] described a stable point known as the Nash Equilibrium. It comprises of those strategy sets where each player opts for the strategy that yields maximum payoff irrespective of the other player's strategy i.e., no player gets an incentive by unilaterally changing her/his strategy. Further, a strategy set of a game is Pareto efficient (or Pareto optimal) if there is no other strategy set that makes atleast one player better off without making any other player worse off. The concept behind this equilibrium point lies in the fact that multiple players contest in a game, and each player's payoff depends on the other players' choice of strategy or decision. Thus, Nash equilibrium is very useful in analysing decision making in situations of war or dilemma.

Table 1 shows the payoff matrix table for a Prisoners' Dilemma game [52]. The game represents a scenario where two prisoners are suspected of committing a crime and are being interrogated in two separate cells. They can either cooperate by accepting their crime or defect by denying their crime. When both accept their crime they get an equal payoff of -1 each. When both deny their crime, they get an equal payoff of -2 each. Further, if one prisoner accepts his crime and the other denies, then the one who denies gets 0 payoff and the one who accepts gets a lower payoff of -3 . In the payoff matrix, the rows represent the strategies of Prisoner 1, and the columns represent the strategies of Prisoner 2. The numbers in each rectangle represent payoffs of prisoners depending on the strategies opted. For example, $(-3,0)$ shows that Prisoner 1 receives a payoff of -3 and Prisoner 2 receives a payoff of 0 for opting for strategy set cooperate and defect, respectively. Analyzing the payoff matrix, we can observe that each player is at a better position by denying his crime, independent of what the other player or prisoner chooses to do. Therefore, both prisoners denying their crime collectively forms the Nash equilibrium of the dilemma game. However, the common welfare/ pareto-optimal move for the prisoners would be cooperation from both players so that they get higher payoff each $(-1)$ as compared to the defection move from both players $(-2)$. Cooperation from both players is pareto-optimal whereas the obtained Nash equilibrium (defection from both players) is not. This contrasting situation is the dilemma in the game, henceforth justifying the role of finding Nash equilibrium in game theory.

There can be two types of Nash Equilibria: pure strategy Nash equilibrium or mixed strategy Nash equilibrium [47]. An example of pure strategy Nash equilibrium is the one discussed above in case of Prisoners' Dilemma game [52]. In mixed strategies, players choose a probability distribution over the set of actions or strategies [61, 47]. Apart from this, there can be games with multiple pure strategy Nash equilibria [23]. If the equilibria comprises of same or corresponding strategies by the players, then the game belongs to the category of coordination games. For instance, battle of the sexes game [23] represented in Table 4 is an example of coordination game. On the other hand, when the equilibria comprises of different or anti-corresponding strategies, the game is an anti-coordination game. An example of anti-coordination game is the hawk-dove game or chicken game [39] as described in Table 6.

## 3 CHSH game

Clauser, Horne, Shimony, and Holt described the CHSH inequality [3] under the assumption of local realism. This inequality can be used to ascertain the presence or absence of quantum
correlations in an underlying quantum system [2]. Although all pure entangled two-qubit states violate the Bell-CHSH inequality confirming the signature of nonlocality, there are mixed states which are entangled but still do not violate the Bell-CHSH inequality [41, 42, 45]. Clearly nature cannot be described using local hidden variable theories; and well within the boundaries of entanglement and nonlocality paradigm, there are problems which require much better physical interpretation.

Alternately, to analyse the nonlocal correlations in a simpler way, the situation can also be portrayed in terms of a game [62], termed as a CHSH game. In order to discuss the results obtained in this article, we first briefly describe the CHSH game. The CHSH game is usually played by two cooperating players; Alice and Bob. In the settings of a game, a referee generates two independent random bits: ' $x$ ' and ' $y$ ', and sends them to Alice and Bob, respectively. These random bits act as inputs to the players. On receiving these input bits, Alice and Bob output their answer bits as ' $a$ ' and ' $b$ ', respectively. Both players win the game if the addition modulo 2 (or XOR) of their outputs is equal to the logical AND of their inputs, i.e., $a \oplus b=x \cdot y$. Alice and Bob both aim at increasing their chances of win, and hence decide a priory the strategy to be used during the game. However, they cannot communicate after the commencement of the game, and do not have prior information about each other's input or output. The only information they have is about their individual inputs (' $x$ ' is known to Alice and ' $y$ ' is known to Bob), based on which they produce their outputs ' $a$ ' and ' $b$ ', respectively. Here, for simplicity, we assume that the probability of an input to take value 0 or 1 is equiprobable. Considering the winning condition of the game, one can easily evaluate that classically the game can be won with utmost $75 \%$ probability. There can be 16 different classical strategies that a player can opt for, out of which 8 strategies give $75 \%$ winning probability. For example, the classical strategy: $y_{A}=\overline{x_{A}}$ and $y_{B}=x_{B}$ of Alice and Bob, respectively result in a win in $\frac{3}{4}$ cases.

On the other hand, quantum mechanics allows Alice and Bob to share a two-qubit entangled state, i.e.,

$$
\begin{equation*}
|\psi\rangle_{A B}=\frac{1}{\sqrt{2}}\left[|0\rangle_{A}|0\rangle_{B}+|1\rangle_{A}|1\rangle_{B}\right] \tag{1}
\end{equation*}
$$

Alice and Bob perform spin-projection measurements on their respective qubits according to the input received, and note the measurement outcomes as output. For example, let us assume that Alice and Bob perform measurements in a general basis $v_{i}(\theta)$ denoted as

$$
\begin{align*}
\left|v_{0}(\theta)\right\rangle & =\cos \theta|0\rangle+\sin \theta|1\rangle \\
\left|v_{1}(\theta)\right\rangle & =-\sin \theta|0\rangle+\cos \theta|1\rangle \tag{2}
\end{align*}
$$

where Alice uses $\theta_{A 0}\left(\theta_{A 1}\right)$ when input ' $x$ ' is $0(1)$, and Bob use $\theta_{B 0}\left(\theta_{B 1}\right)$ when input ' $y$ ' is 0 (1). Depending on the measurement outcomes as $\left|v_{0}\right\rangle$ or $\left|v_{1}\right\rangle$, Alice and Bob announce their outputs as 0 or 1 , respectively. As per the rules of the game defined above, when product of inputs is 0 , then the game will lead to success if and only if the outputs also have the same value. Thus, with the above defined quantum strategy, the winning probability of the game at different values of ' $x$ ' and ' $y$ ' is shown in Table I. Therefore, the total winning probability of the game is $\frac{1}{4}\left[\cos ^{2}\left(\theta_{A 0}-\theta_{B 0}\right)+\cos ^{2}\left(\theta_{A 0}-\theta_{B 1}\right)+\cos ^{2}\left(\theta_{A 1}-\theta_{B 0}\right)+\sin ^{2}\left(\theta_{A 1}-\theta_{B 1}\right)\right]$. For optimizing the winning probability, if Alice and Bob choose $\theta_{A 0}=0, \theta_{A 1}=\frac{\pi}{4}, \theta_{B 0}=\frac{\pi}{8}$, and $\theta_{B 1}=-\frac{\pi}{8}$, the winning probability of the game is $\cos ^{2}\left(\frac{\pi}{8}\right) \approx 0.8535$. Clearly, quantum

Table 2. Dependence of winning prospects on the inputs received by the players in CHSH game

| x | y | Winning probability of CHSH game |
| :---: | :---: | :---: |
| 0 | 0 | $\cos ^{2}\left(\theta_{A 0}-\theta_{B 0}\right)$ |
| 0 | 1 | $\cos ^{2}\left(\theta_{A 0}-\theta_{B 1}\right)$ |
| 1 | 0 | $\cos ^{2}\left(\theta_{A 1}-\theta_{B 0}\right)$ |
| 1 | 1 | $\sin ^{2}\left(\theta_{A 1}-\theta_{B 1}\right)$ |

strategy gives better winning probability (above $85 \%$ ) than the classical winning probability ( $75 \%$ ) in the CHSH game.

## 4 Structure of Bayesian games that holds direct relation with CHSH inequality

The structure of a CHSH game can be utilized in the settings of a Bayesian game [21], where the players are of different types ( ${ }^{6} x_{A}$ ' and ' $x_{B}$ ') depending on the inputs (' $x$ ' and ' $y$ ') they receive from the referee. For instance, input $x=0$ corresponds to type 0 of Alice, i.e., $x_{A}=0$; input $x=1$ corresponds to type 1 of Alice, i.e., $x_{A}=1$; input $y=0$ corresponds to type 0 of Bob, i.e., $x_{B}=0$; and input $y=1$ corresponds to type 1 of Bob, i.e., $x_{B}=1$. Moreover, outputs (' $a$ ' and ' $b$ ') define the strategies (' $y_{A}$ ' and ' $y_{B}$ ') that the players opt for. Thus, in order to maintain the structure of a Bayesian game similar to the winning conditions of a CHSH game, when $x_{A}=x_{B}=0$, or $x_{A} \neq x_{B}$, Alice and Bob get a non-zero payoff on choosing strategies $\left(y_{A}\right.$ and $\left.y_{B}\right)$ such that $y_{A} \oplus_{2} y_{B}=0$. Similarly, to satisfy the CHSH setting, when $x_{A}=x_{B}=1$, the players get a non-zero payoff on choosing strategies $y_{A}\left(y_{B}\right)=0(1)$ or $y_{A}\left(y_{B}\right)=1(0)$. Hence, the overall condition of win in a CHSH game $\left(x_{A} \cdot x_{B}=y_{A} \oplus_{2} y_{B}\right)$ enables quantum players to exploit nonlocal correlations existing in the shared quantum system in order to win the game. Table 3 shows the payoffs attained by different types of Alice and Bob in a game with the above defined settings. In each cell, the first number represents the payoff of Player 1, i.e., Alice, and the second number represents the payoff of Player 2, i.e., Bob. The diagonal and off-diagonal terms appearing in the two payoff matrices are results of winning conditions of the game.

Table 3. Payoff of Alice and Bob in a general game setting where dependence of payoff on type of player commensurate with the input-output relation in CHSH game (Here, $u_{1}^{A}, u_{1}^{B}, u_{2}^{A}, u_{2}^{B}, u_{3}^{A}$, $u_{3}^{B}, u_{4}^{A}$, and $u_{4}^{B}$ are non-zero)

| Alice | Bob | $y_{B}=0$ |
| :---: | :---: | :---: |
| $y_{B}=1$ |  |  |
| $y_{A}=0$ | $u_{1}^{A}, u_{1}^{B}$ | 0,0 |
| $y_{A}=1$ | 0,0 | $u_{2}^{A}, u_{2}^{B}$ |

(a) $x_{A} \cdot x_{B}=0$

| Alice Bob | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | 0,0 | $u_{3}^{A}, u_{3}^{B}$ |
| $y_{A}=1$ | $u_{4}^{A}, u_{4}^{B}$ | 0,0 |

(b) $x_{A} \cdot x_{B}=1$

## 5 Combination of Coordination and Anti-coordination games

Coordination and anti-coordination games are those in which there is no fixed dominant strategy for a player. The players should coordinate (or anti-coordinate) in order to attain maximum payoff for themselves, in a respective game. Coordination means that players choose same or corresponding strategies at equilibria. On the other hand, anti-coordination signifies that the players choose different strategies or strategies different from the corresponding ones,
at equilibria. Hence both games have multiple Nash equilibria. In order to maintain the setup of a game defined in Table 3, the players should coordinate and choose same strategies at equilibrium, when the logical AND of the type of players ( $x_{A}$ and $x_{B}$ ) is 0 ; and should anti-coordinate and choose different strategies at equilibrium, when the logical AND of the type of players $\left(x_{A}\right.$ and $\left.x_{B}\right)$ is 1 .

### 5.1 Combination of a conflicting interest (Battle of the Sexes game) and a common interest game

Battle of the Sexes (BoS) game is a two-player coordination game, where two players, a husband and a wife wish to spend an evening together. However they have different choices about spending time together. The wife prefers to watch a movie whereas the husband prefers to watch a football match. Table 4 shows the payoffs of husband and wife in the game. One can see that there is no fixed dominating strategy for any player. Still, if husband

Table 4. Battle of the Sexes game

| Husband Wife | Football | Movie |
| :---: | :---: | :---: |
| Football | $1, \frac{1}{2}$ | 0,0 |
| Movie | 0,0 | $\frac{1}{2}, 1$ |

watches a football match, then wife gets a better payoff by opting for the same. Similarly, if wife watches a movie, then husband gets a better payoff by opting to go for the movie. Thus, $\{$ Football, Football $\}$ and $\{$ Movie, Movie $\}$ are two pareto-optimal Nash equilibrium of the game. Further, the game is a conflicting interest one, since husband prefers the $\{$ Football, Football $\}$ equilibrium, and wife prefers the $\{$ Movie, Movie $\}$ equilibrium. In contrast the common interest games are those in which both players benefit equally by opting for a Nash equilibrium strategy, and do not have any preference over the other [23].

For our purpose, we have combined BoS game with a common interest anti-coordination game to demonstrate the role of quantum strategies in Bayesian games with CHSH-type dependence on payoffs. When the logical AND of the type of players is 0 , then the players play the coordination BoS game and when the logical AND of the type of players is 1 , then the players play an anti-coordination game, the payoffs of which are defined in Table 5. Here,

Table 5. Payoff of Alice and Bob when they either play a conflicting interest coordination game similar to Battle of Sexes game or a common interest anti-coordination game

| Alice Bob | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | $\frac{16}{9}, \frac{8}{9}$ | 0,0 |
| $y_{A}=1$ | 0,0 | $\frac{8}{9}, \frac{16}{9}$ |

(a) $x_{A} \cdot x_{B}=0$

| Alice | Bob | $y_{B}=0$ |
| :---: | :---: | :---: |
| $y_{B}=1$ |  |  |
| $y_{A}=0$ | 0,0 | $\frac{4}{3}, \frac{4}{3}$ |
| $y_{A}=1$ | $\frac{4}{3}, \frac{4}{3}$ | 0,0 |

(b) $x_{A} \cdot x_{B}=1$
we elaborate the usefulness of general two-qubit pure Bell states as well as mixed quantum states (Werner and Horodecki states) instead of a maximally entangled state [24] when the quantum strategies correspond to performing different single-parameter measurements on the qubits.

### 5.1.1 Classical scenario

There can be 16 different classical strategies since $y_{A}$ and $y_{B}$ can take two different ( 0 or 1 ) values, respectively, for two different ( 0 or 1 ) individual values of $x_{A}$ and $x_{B}$. After analysing all possible classical strategy sets, one can conclude that for no classical strategy, the total payoff of the players exceeds 2 . In addition, there are three Nash equilibria for the game, i.e.,

- $y_{A}=0$ irrespective of the value of $x_{A}$ and $y_{B}=x_{B}$ : This strategy leads to a paretooptimal Nash equilibrium preferred by Alice since Alice gets a payoff of $\frac{11}{9}$ and Bob gets a payoff of $\frac{7}{9}$;
- $y_{A}=x_{A}$ and $y_{B}=\overline{x_{B}}$ : This strategy also leads to a pareto-optimal Nash equilibrium preferred by none of the players since both players get an equal payoff of 1 ; and
- $y_{A}=\overline{x_{A}}$ and $y_{B}=1$ irrespective of the value of $x_{B}$ : This strategy further leads to a pareto-optimal Nash equilibrium preferred by Bob since Alice gets a payoff of $\frac{7}{9}$ and Bob gets a payoff of $\frac{11}{9}$


### 5.1.2 Quantum scenario

However, the situation becomes different when the players are allowed to share an entangled quantum state. Let us assume that the probability of Alice to be of type $0\left(x_{A}=0\right)$ be ' $p$ ' and the probability of Bob to be of type $0\left(x_{B}=0\right)$ be ' $q$ '. By using any quantum strategy thereof, the sum of payoff of Alice $\left(\$_{A}\right)$ and payoff of Bob $\left(\$_{B}\right)$ is given as

$$
\begin{array}{r}
\$_{A}+\$_{B}=\frac{8}{3}\left[p q\left(P_{00}^{00}+P_{00}^{11}\right)+p(1-q)\left(P_{01}^{00}+P_{01}^{11}\right)+(1-p) q\left(P_{10}^{00}+P_{10}^{11}\right)\right. \\
\left.+(1-p)(1-q)\left(P_{11}^{01}+P_{11}^{10}\right)\right] \tag{3}
\end{array}
$$

where probability $P_{i j}^{k l}$ is defined as a product of two conditional probabilities, i.e., $P\left(y_{A}=\right.$ $\left.k \mid x_{A}=i\right) P\left(y_{B}=l \mid x_{B}=j\right)$. In game theory, the notion ' $\$$ ' represents payoff to a player in a game which may be considered as an incentive to the player for using a certain strategy. The payoff may be monetary or more spiritual such as happiness quotient. If there are two players Alice and Bob in the game then ' $\$_{A}$ ' and ' $\${ }_{B}$ ' represents payoff of Alice and Bob in the game. Further, the coefficient $\frac{8}{3}$ in Eq. (3) appears because of the values assumed as payoffs of players in Table 5. Moreover, the structure of the designed game requires that the sum of payoffs of both players in the games in Table 5 (a) and Table 5 (b) should be same, just as in our case, i.e., $\frac{16}{9}+\frac{8}{9}=\frac{4}{3}+\frac{4}{3}=\frac{8}{3}$. Since the game in Table 5 (a) is a payoff representation of Battle-of-Sexes game, it must obey certain conditions such as, $\$_{A}>\$_{B}$ for strategies $y_{A}=0$ and $y_{B}=0$, and $\$_{A}<\$_{B}$ for strategies $y_{A}=1$ and $y_{B}=1$. The coefficient term could be any non-negative number. The payoff values are taken as an example of a Bayesian game combining a conflicting interest coordination game and a common interest anti-coordination game. Assuming the strategies $y_{A}=0$ and $y_{B}=0$ correspond to measurement outcomes yielding positive eigenvalue $(+1)$, and strategies $y_{A}=1$ and $y_{B}=1$ correspond to measurement outcomes yielding negative eigenvalue $(-1)$; the expectation value $E(i j)=E\left(x_{A}=i, x_{B}=j\right)$ can be defined as $P_{i j}^{00}-P_{i j}^{01}-P_{i j}^{10}+P_{i j}^{11}$. Thus the total payoff of players can be re-expressed as

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{4}{3}[1+p q E(00)+p(1-q) E(01)+(1-p) q E(10)-(1-p)(1-q) E(11)] \tag{4}
\end{equation*}
$$

For simplicity, we assume that $p=q=\frac{1}{2}$. Under this assumption, the sum of the payoffs of Alice and Bob hold a direct relation with the Bell-CHSH operator $\langle B\rangle[2]$ as

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{4}{3}\left[1+\frac{\langle B\rangle}{4}\right] \tag{5}
\end{equation*}
$$

where $\$_{A}=\frac{4}{3}\left[\frac{1}{2}+\frac{\langle B\rangle}{8}\right]$ and $\$_{B}=\frac{4}{3}\left[\frac{1}{2}+\frac{\langle B\rangle}{8}\right]$. In order to attain a better payoff in the quantum scenario in comparison to classical scenario, we consider that Alice and Bob share a general two-qubit entangled quantum state. If $x_{A}$ and $x_{B}$ are types of Alice and Bob, respectively, and $y_{A}$ and $y_{B}$ are their respective moves in the game, then the measurements performed by them as a team on their qubits can be represented as $A_{x_{A}}^{y_{A}}$ and $B_{x_{B}}^{y_{B}}$ [24], where

$$
\begin{align*}
A_{0}^{a} & =\left|\Phi_{a}(0)\right\rangle\left\langle\Phi_{a}(0)\right|, & A_{1}^{a}=\left|\Phi_{a}\left(\frac{\pi}{4}\right)\right\rangle\left\langle\Phi_{a}\left(\frac{\pi}{4}\right)\right| \\
B_{0}^{b} & =\left|\Phi_{b}(\lambda)\right\rangle\left\langle\Phi_{b}(\lambda)\right|, & B_{1}^{b}=\left|\Phi_{b}(-\lambda)\right\rangle\left\langle\Phi_{b}(-\lambda)\right| \tag{6}
\end{align*}
$$

The basis of measurement $\left|\phi_{0}(\theta)\right\rangle$ and $\left|\phi_{1}(\theta)\right\rangle$ are the same as $\left|v_{0}(\theta)\right\rangle$ and $\left|v_{1}(\theta)\right\rangle$ respectively, as defined in Eq. (2).

Moreover, one can also relate this quantum strategy with the experimental settings in a Bell-CHSH experiment $[2,3]$. For example, in a Bell-CHSH experimental set-up, Alice randomly chooses to perform a measurement $Q=\left[\begin{array}{cc}\cos \theta_{1} & \sin \theta_{1} e^{-i \phi_{1}} \\ \sin \theta_{1} e^{i \phi_{1}} & -\cos \theta_{1}\end{array}\right]$ or $R=$ $\left[\begin{array}{cc}\cos \theta_{1}^{\prime} & \sin \theta_{1}^{\prime} e^{-i \phi_{1}^{\prime}} \\ \sin \theta_{1}^{\prime} e^{i \phi_{1}^{\prime}} & -\cos \theta_{1}^{\prime}\end{array}\right]$ and similarly Bob also randomly chooses to perform a measurement $S=\left[\begin{array}{cc}\cos \theta_{2} & \sin \theta_{2} e^{-i \phi_{2}} \\ \sin \theta_{2} e^{i \phi_{2}} & -\cos \theta_{2}\end{array}\right]$ or $T=\left[\begin{array}{cc}\cos \theta_{2}^{\prime} & \sin \theta_{2}^{\prime} e^{-i \phi_{2}^{\prime}} \\ \sin \theta_{2}^{\prime} e^{i \phi_{2}^{\prime}} & -\cos \theta_{2}^{\prime}\end{array}\right]$, on their respective qubits. The expectation value of Bell-CHSH operator thus designed is equal to $E(Q S)+$ $E(R S)+E(R T)-E(Q T)$ which is same as the expectation value of Bell-CHSH operator $\langle B\rangle$ obtained in Eq. (5). However, the measurements performed by Alice and Bob as quantum strategies in the game (Eq. (6)) correspond to the measurements in the experimental set-up when $\theta_{1}=0, \theta_{1}^{\prime}=\frac{\pi}{2}, \theta_{2}=2 \lambda, \theta_{2}^{\prime}=-2 \lambda$, and $\phi_{1}=\phi_{1}^{\prime}=\phi_{2}=\phi_{2}^{\prime}=0$. This can also be termed as restricted one parameter $(\lambda)$ measurements. We choose the above set of measurements so as to achieve maximum expectation value of the Bell-CHSH operator for a general two-qubit Bell state. Nevertheless, we can still witness quantum advantage in a wide class of pure and mixed states.

We now proceed to analyse the total payoff of players in the game. For this, we consider that Alice and Bob share a general two-qubit Bell state given by

$$
\begin{equation*}
|\phi\rangle_{\text {Bell }}=\cos \theta|00\rangle+\sin \theta|11\rangle \tag{7}
\end{equation*}
$$

Clearly, the state in Eq. (7) violates the Bell-CHSH inequality for all values of $\theta \epsilon\left(0, \frac{\pi}{4}\right]$, and the violation increases with the increase in degree of entanglement of the shared state, i.e, the Bell-CHSH inequality is maximally violated by the maximally entangled state, and not by a non-maximally entangled state. The total payoff of both players sharing a general two-qubit Bell state can be evaluated as

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{2}{3}[2+\sin 2 \theta \sin 2 \lambda+\cos 2 \lambda] \tag{8}
\end{equation*}
$$



Fig. 1. Combination of BoS and common interest anti-coordination game when different quantum states are shared among the players

One can see that the same value of total payoff can be achieved when players share a twoqubit arbitrary state $\left|\psi_{\text {arbitrary }}\right\rangle=\cos \theta|00\rangle+e^{i \phi} \sin \theta|11\rangle$, and perform measurements given in Eq. (6) in an arbitrary orthogonal basis given by $\left|\phi_{0}(\theta)\right\rangle=\cos \theta|0\rangle+e^{i \phi / 2} \sin \theta|0\rangle$ and $\left|\phi_{1}(\theta)\right\rangle=-e^{-i \phi / 2} \sin \theta|0\rangle+\cos \theta|1\rangle$. Moreover, it can be evaluated that the maximum value of the summed payoff in Eq. (8) is achieved at $\lambda=\frac{1}{2} \tan ^{-1}(\sin 2 \theta)$. Thus, Figure (1) shows that the sum of payoffs of Alice and Bob thereof always exceeds the classical bound/ average classical payoff of 2 . In other words, general two-qubit pure Bell states offer quantum advantage for all values of state parameter $\theta$ in a CHSH-type Bayesian game setting including conflicting and common interest games. Therefore, although the maximally entangled Bell state gives maximum total payoff of 2.276 , non-maximally entangled Bell states for $0<\theta<\frac{\pi}{4}$ also give better total payoff than any other classical strategy. We have used concurrence $[63,64,65]$ as a measure of the degree of two-qubit entanglement throughout the article. For a general mixed state $\rho$, it is defined as

$$
\begin{equation*}
C(\rho)=\max \left(0, \lambda_{1}-\lambda_{2}-\lambda_{3}-\lambda_{4}\right) \tag{9}
\end{equation*}
$$

where $\lambda_{i}$ are the singular values of $\sqrt{\rho} \sqrt{\widetilde{\rho}}$ with $\widetilde{\rho}=\left(\sigma_{y} \otimes \sigma_{y}\right) \rho^{*}\left(\sigma_{y} \otimes \sigma_{y}\right)$; and $\lambda_{k}>\lambda_{k+1}$. Concurrence of the general two-qubit Bell state defined in Eq. (7) is $\sin 2 \theta$.

As examples of mixed states for this scenario, we consider Werner class states [41] and Horodecki states [42]. We first describe the Werner states as a linear combination of a Bell state $\left|\phi^{+}\right\rangle=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ and a maximally mixed state $I_{4}=|00\rangle\langle 00|+|01\rangle\langle 01|+|10\rangle\langle 10|+$ $|11\rangle\langle 11|$ as given below in Eq. (10).

$$
\begin{equation*}
\rho_{w e r n e r}=\gamma\left|\phi^{+}\right\rangle\left\langle\phi^{+}\right|+\frac{1-\gamma}{4} I_{4} \tag{10}
\end{equation*}
$$

where $\gamma$ is a state parameter. Acin et al. [66] have shown that Werner states violate the Bell-CHSH inequality only for $\gamma>\frac{1}{\sqrt{2}}$, i.e., even though they possess entanglement for $\frac{1}{3}<\gamma \leq \frac{1}{\sqrt{2}}$, they violate the inequality only for states having concurrence (C) more than $\frac{3}{2 \sqrt{2}}-\frac{1}{2}$. Moreover, the total payoff of both the players sharing a Werner state is

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{2}{3}[2+\gamma \sin 2 \lambda+\gamma \cos 2 \lambda] \tag{11}
\end{equation*}
$$

Clearly the maximum value of the summed payoff in Eq. (11) is achieved at $\lambda=\frac{\pi}{8}$. Thus, Figure (1) shows that the sum of payoffs of Alice and Bob thereof exceeds the classical bound/ average classical payoff of 2 for $\gamma>\frac{1}{\sqrt{2}}$ or $C=(1.5 \gamma-0.5)>0.5606$. In other words, Werner states offer quantum advantage for a fixed range of state parameters only, and for $\gamma \leq \frac{1}{\sqrt{2}}$, classically defined strategies yield a better payoff than a team of Alice and Bob equipped with quantum strategies.

As mentioned above, another class of mixed states that we analyse are Horodecki states which are defined as a linear combination of a Bell state $\left|\psi^{+}\right\rangle=\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)$ and a
separable state $|00\rangle$ as shown below in Eq. (12)

$$
\begin{equation*}
\rho_{\text {horodecki }}=\mu\left|\psi^{+}\right\rangle\left\langle\psi^{+}\right|+(1-\mu)|00\rangle\langle 00| \tag{12}
\end{equation*}
$$

where $\mu$ is the state parameter. It was established that Horodecki states violate the BellCHSH inequality for $\mu>\frac{1}{\sqrt{2}}$, i.e., even though they possess entanglement for all values of the state parameter, they violate the Bell-CHSH inequality only for states having concurrence greater than $\frac{1}{\sqrt{2}}[67]$. Thus, the total payoff of both the players sharing a Horodecki state is

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{2}{3}[2+\mu \sin 2 \lambda+(1-2 \mu) \cos 2 \lambda] \tag{13}
\end{equation*}
$$

As in the previous cases, one can calculate the maximum value of the summed payoff in Eq. (13) which is achieved at $\lambda=\frac{1}{2} \tan ^{-1} \frac{\mu}{1-2 \mu}$. Therefore, Figure (1) demonstrates that the sum of payoffs of Alice and Bob thereof exceeds the classical bound of 2 for $C(=\mu)>0.8$. In other words, Horodecki states offer quantum advantage for a fixed range of state parameter only. Interestingly, for the range $\frac{1}{\sqrt{2}}<C(=\mu)<0.8$, even though Horodecki states violate the Bell-CHSH inequality, they do not provide any quantum advantage in the designed set-up of the game. This can be attributed to the difference in measurement settings of our game from the Bell-CHSH experimental set-up, as explained above.

Hence, any general two-qubit pure Bell state, any Werner class for $\gamma>\frac{1}{\sqrt{2}}$, or any Horodecki state for $\mu>0.8$, can be used as a quantum resource by Alice and Bob in order to exceed the total payoff from the classical bound of 2 . In other words, we have analysed a large set of pure and mixed states for which the above defined quantum strategy gives better summed payoff than any strategy opted by classical players. Thus, we can conclude that just like pure states, mixed quantum states are also useful for the players in the game described above. However, for a fixed value of concurrence, pure states help attain better payoff in the game than mixed states.

### 5.2 Combination of two conflicting interest games (Battle of the Sexes game and Chicken game)

In the previous subsection, we analysed the scenario where players either got engaged in a conflicting interest game or a common interest game, depending on their inputs. In this section, we attempt to combine two conflicting interest games to understand the benefits of nonlocality when the players always have conflicting interests. A simple example of conflicting interest anti-coordination game, also known as the hawk-dove game or snowdrift game, is Chicken game [39]. Here, we first briefly describe Chicken game through two drivers who drive towards each other, and can suffer a head-on collision if both keep driving straight. On the other hand, if one driver swerves and the other does not, the one who swerved will be named "chicken", signifying him as a coward. Therefore, the coward gets a lower payoff than the one who drives straight. However, the lowest payoff incurs when none of the players risk to be a chicken or a coward, but rather choose to go straight, i.e., if both drivers do not swerve. Thus, there is no dominant strategy for this game as indicated in Table 6.

Table 6. Chicken game

| Driver 1 Driver 2 | Swerve | Straight |
| :---: | :---: | :---: |
| Swerve | 0,0 | $-1,+1$ |
| Straight | $+1,-1$ | $-5,-5$ |

We can see that if Driver 1 goes straight, then Driver 2 gets a better payoff by opting to swerve rather than going straight. Similarly, if Driver 2 goes straight, then Driver 1 gets a better payoff by opting to swerve instead of driving straight. Thus $\{$ Swerve, Straight $\}$ and $\{$ Straight, Swerve\} are two pareto-optimal Nash equilibria of the game. Further, the game is a conflicting interest game as Driver 1 prefers the $\{$ Straight, Swerve $\}$ equilibrium, and Driver 2 prefers the $\{$ Swerve, Straight $\}$ equilibrium. Due to no fixed dominating strategy, the game can be termed as an anti-coordination one. Being a conflicting interest anti-coordination game involving greed between the players, but no fear, analysing the effects of quantum strategies on such games becomes an interesting problem. Thus, we have combined BoS game with Chicken game to demonstrate the role of quantum strategies in conflicting interest Bayesian games with CHSH-type dependence on payoffs. For this, we consider that when the logical AND of the type of players is 0 , then the players play the coordination $\operatorname{BoS}$ game and when the logical AND of the type of players is 1 , then the players play the anti-coordination Chicken game; the payoffs of which are defined in Table 7.

Table 7. Payoff of Alice and Bob when they either play a conflicting interest coordination game similar to Battle of Sexes game or a conflicting interest anti-coordination game similar to Chicken game

| Alice | $y_{B}=0$ | $y_{B}=1$ |  |
| :---: | :---: | :---: | :---: |
| $y_{A}=0$ | $\frac{4}{3}, \frac{2}{3}$ | 0,0 |  |
| $y_{A}=1$ | 0,0 | $\frac{2}{3}, \frac{4}{3}$ |  |
| (a) $x_{A} \cdot x_{B}=0$ |  |  |  |


| Alice | Bob | $y_{B}=0$ |
| :---: | :---: | :---: |
| $y_{B}=1$ |  |  |
| $y_{A}=0$ | 0,0 | $-\frac{1}{2}, \frac{5}{2}$ |
| $y_{A}=1$ | $\frac{5}{2},-\frac{1}{2}$ | $-1,-1$ |

(b) $x_{A} \cdot x_{B}=1$

### 5.2.1 Classical scenario

There are three Nash equilibrium for the game, i.e.,

- $y_{A}=0$ irrespective of the value of $x_{A}$ and $y_{B}=x_{B}$ : This strategy leads to a paretooptimal Nash equilibrium preferred by Bob since Alice gets a payoff of $\frac{13}{24}$ and Bob gets a payoff of $\frac{23}{24}$;
- $y_{A}=x_{A}$ and $y_{B}=\overline{x_{B}}$ : This strategy also leads to a pareto-optimal Nash equilibrium preferred by Alice since Alice gets a payoff of $\frac{9}{8}$ and Bob gets a payoff of $\frac{3}{8}$; and
- $y_{A}=1$ and $y_{B}=1$ irrespective of the value of $x_{A}$ and $x_{B}$ : This strategy leads to non pareto-optimal Nash equilibrium since Alice gets a payoff of $\frac{1}{4}$ and Bob gets a payoff of $\frac{3}{4}$.

For the above defined game, the maximum total payoff achieved by opting any of the two pareto-optimal Nash equilibrium strategies is $\frac{3}{2}$, and by opting for the non pareto-optimal

Nash equilibrium strategies is 1 . Thus, the average total classical NE payoff is $\frac{1}{3}\left[\frac{3}{2}+\frac{3}{2}+1\right]=$ $\frac{4}{3}$.

### 5.2.2 Quantum scenario

Unlike the previous game setting, where a conflicting interest game is combined with a common interest game, the combination of two conflicting interest games lead to some interesting observations. Similar to the previous case, let us start with assuming the probability of Alice to be of type $0\left(x_{A}=0\right)$ be ' $p$ ' and the probability of Bob to be of type $0\left(x_{B}=0\right)$ be ' $q$ '. By using any quantum strategy thereof, the sum of payoffs of Alice and Bob is given as

$$
\begin{array}{r}
\$_{A}+\$_{B}=2\left[p q\left(P_{00}^{00}+P_{00}^{11}\right)+p(1-q)\left(P_{01}^{00}+P_{01}^{11}\right)+(1-p) q\left(P_{10}^{00}+P_{10}^{11}\right)+\right. \\
\left.(1-p)(1-q)\left(P_{11}^{01}+P_{11}^{10}-P_{11}^{11}\right)\right] \tag{14}
\end{array}
$$

where the probability $P_{i j}^{k l}$ are defined earlier. The occurrence of the factor 2 in Eq. (14) can also be understood following the description of previous game setting. Further, considering the strategies $y_{A}=0$ and $y_{B}=0$ correspond to measurement outcomes yielding positive eigenvalue $(+1)$ and strategies $y_{A}=1$ and $y_{B}=1$ correspond to measurement outcomes yielding negative eigenvalue ( -1 ), the expectation value $E(i j)=E\left(x_{A}=i, x_{B}=j\right)$ can be defined as $P_{i j}^{00}-P_{i j}^{01}-P_{i j}^{10}+P_{i j}^{11}$. Thus Eq. (14) is re-expressed as

$$
\begin{array}{r}
\$_{A}+\$_{B}=2\left[\frac{1}{2}+\frac{1}{2}\{p q E(00)+p(1-q) E(01)+(1-p) q E(10)-(1-p)(1-q) E(11)\}-\right. \\
\left.(1-p)(1-q) P_{11}^{11}\right] \tag{15}
\end{array}
$$

As earlier, for simplicity, we assume that $p=q=\frac{1}{2}$. Therefore, Eq. (14) can be re-expressed in terms of the Bell-CHSH operator $\langle B\rangle[2]$ along with an additional conditional probability term as

$$
\begin{equation*}
\$_{A}+\$_{B}=2\left[\frac{1}{2}+\frac{\langle B\rangle}{8}-\frac{P_{11}^{11}}{4}\right] \tag{16}
\end{equation*}
$$

where $\$_{A}=2\left[\frac{7}{24}+\frac{\langle B\rangle}{16}-\frac{P_{11}^{11}}{8}\right]$ and $\$_{B}=2\left[\frac{5}{24}+\frac{\langle B\rangle}{16}-\frac{P_{11}^{11}}{8}\right]$. Since we are describing the game under CHSH setting, it is important to note that a maximally entangled pure state will violate the Bell-CHSH inequality maximally. Furthermore, since $P_{11}^{11}>0$ and classically $\langle B\rangle_{\max }=2$, we can replace these values in Eq. (16) and verify that the classical bound for this game is $\leq \frac{3}{2}$.

Clearly, when players share a general two-qubit Bell state as given by Eq. (7) and perform measurements defined in Eq. (6), then the total payoff of both players is

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{1}{8}[7+5 \sin 2 \theta \sin 2 \lambda+(\cos 2 \theta+4) \cos 2 \lambda] \tag{17}
\end{equation*}
$$

As explained earlier, one can also attain the same payoff when players share a two-qubit arbitrary state $\left|\psi_{\text {arbitrary }}\right\rangle=\cos \theta|00\rangle+e^{i \phi} \sin \theta|11\rangle$, and perform measurements given in Eq. (6) in an arbitrary orthogonal basis given by $\left|\phi_{0}(\theta)\right\rangle=\cos \theta|0\rangle+e^{i \phi / 2} \sin \theta|0\rangle$ and $\left|\phi_{1}(\theta)\right\rangle=$ $-e^{-i \phi / 2} \sin \theta|0\rangle+\cos \theta|1\rangle$.

Therefore, the maximum value of the summed payoff in Eq. (17) is achieved at $\lambda=$ $\frac{1}{2} \tan ^{-1} \frac{5 \sin 2 \theta}{\cos 2 \theta+4}$. Evidently, Figure (2) shows that the sum of payoffs of Alice and Bob


Fig. 2. Combination of $\operatorname{BoS}$ and Chicken game when different quantum states are shared among the players
sharing a general two-qubit Bell state always exceeds the classical bound of $\frac{3}{2}$ and average classical Nash equilibrium payoff of $\frac{4}{3}$. In other words, general two-qubit pure Bell states offer quantum advantage for all values of the state parameter $\theta$. Surprisingly, the maximally entangled Bell state $\left(\theta=\frac{\pi}{4}\right)$ does not give maximum total payoff. However, a non-maximally entangled state at $\theta=40.188^{\circ}$ with concurrence 0.986 gives maximum summed payoff of approx. 1.6819 for both players. This is contrary to our general belief that the maximally entangled pure state will always be more efficient than a non-maximally entangled pure state, at least under the setting of the discussed game. Although, the value of Bell-CHSH operator $\langle B\rangle$ is maximum at $\theta=45^{\circ}$, the value of $\langle B\rangle-2 P_{11}^{11}$ at $\theta=40.188^{\circ}$ is more than the value of $\langle B\rangle-2 P_{11}^{11}$ at $\theta=45^{\circ}$ while maximizing total payoff $\$_{A}+\$_{B}$. Since $P_{11}^{11}=P\left(y_{A}=1, y_{B}=\right.$ $1 \mid x_{A}=1, x_{B}=1$ ) is a very small number, an angle very close to $45^{\circ}$ (with concurrence very close to 1) gives the maximum total payoff. Furthermore, the analysis shows that all non-maximally entangled Bell states in the range $0<\theta \leq \frac{\pi}{4}$ give better total payoff than any classical strategy. Interestingly, non-maximally entangled states for $34.1804^{\circ} \leq \theta<45^{\circ}$ lead to a better payoff in the game in comparison to a maximally entangled Bell state.

As examples of entangled mixed states, we again consider Werner and Horodecki states as defined in Eq. (10) and Eq. (12), respectively. When the players share a mixed Werner class
state, then the total payoff of both players is

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{1}{8}[7+5 \gamma \sin 2 \lambda+4 \gamma \cos 2 \lambda] \tag{18}
\end{equation*}
$$

Thus, the maximum value of the summed payoff in Eq. (18) is achieved at $\lambda=\frac{1}{2} \tan ^{-1} \frac{5}{4}$. Figure (2) describes that the sum of payoffs of Alice and Bob sharing a Werner state exceed the classical bound of $\frac{3}{2}$ for $C(=1.5 \gamma-0.5)>0.6712$ and average classical Nash equilibrium payoff of $\frac{4}{3}$ for $C>0.3589$. Similar to the previous case, a mixed Werner class state offers quantum advantage for a fixed range of state parameter only. However, unlike the previous case, even though Werner states do not exhibit non-local correlations for $0.5726<\gamma<\frac{1}{\sqrt{2}}$, i.e. $0.3589<C<0.5606$, they still provide advantage to quantum players in this particular game setting. Interestingly, Werner states provide advantage to quantum players over classical players in the above region even though they do not violate the Bell-CHSH inequality in that particular region is attributed to the combination of two conflicting interest games in comparison to the previous case (combination of a conflicting interest game with a common interest game) where no such result was obtained.

Similarly, when the players share a Horodecki state, then the total payoff of both players is

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{1}{8}[7+5 \mu \sin 2 \lambda+(5-9 \mu) \cos 2 \lambda] \tag{19}
\end{equation*}
$$

One can evaluate that the maximum value of the summed payoff in Eq. (19) is achieved at $\lambda=\frac{1}{2} \tan ^{-1} \frac{5 \mu}{5-9 \mu}$. Thus, Figure (2) shows that the sum of payoffs of Alice and Bob sharing a Horodecki state exceeds the classical bound of $\frac{3}{2}$ for $C(=\mu)>0.85$ only. However, the sum of payoffs of Alice and Bob exceeds the average classical Nash equilibrium payoff of $\frac{4}{3}$ for $C<0.1576$ and $C>0.6913$. This result is gives an interesting indication that the players get a better payoff on sharing a Horodecki states in the range of state parameter, i.e., $C(=\mu)<\frac{1}{\sqrt{2}}$, even though the quantum state does not possess non-local correlations in that range. The states however possess entanglement (non-zero concurrence), which is again visible in the plots. Clearly, the results obtained for combination of two conflicting interest games are quite different and interesting from the case where we consider the combination of a common interest and a conflicting interest game.

This study which highlights the advantages offered by non-maximally entangled pure states in comparison to the maximally entangled pure state due to the terms such as $P_{11}^{11}$ will definitely motivate us to study games where such contribution to total payoff may have more drastic effect towards the final output of the game. In addition the analysis also highlights that under the proposed game settings use of mixed entangled states may also result in advantage even if the state did not violate the Bell-CHSH inequality. Moreover, it further paves way to study the important property of randomness in quantum states, which we talk about in the next section.

## 6 Representation of tilted Bell-type inequality in a Bayesian game setting

Like nonlocal correlations, randomness is also inherent to the foundations of quantum mechanics. The outcomes of an experiment designed to test the Bell inequality formalism are
always random (for violation of the inequality), i.e., measurement outcomes cannot be deterministically predicted within quantum theory. Acin et. al [15] have shown that the maximal violation of the Bell-CHSH inequality involves the generation of only 1.23 bits of randomness instead of generating 2 bits of global randomness. Interestingly, they found that less entangled states can produce randomness close to 2 bits, showing that there is no direct relation between entanglement (or nonlocality) with randomness. In order to capture high randomness in non-maximally entangled states, a specific class of Bell-type inequality was defined, i.e.,

$$
\begin{equation*}
I_{\alpha}^{\beta}:=\beta\left\langle A_{0}\right\rangle+\alpha\left\langle A_{0} B_{0}\right\rangle+\alpha\left\langle A_{0} B_{1}\right\rangle+\left\langle A_{1} B_{0}\right\rangle-\left\langle A_{1} B_{1}\right\rangle \geq \beta+2 \alpha \tag{20}
\end{equation*}
$$

The inequality in Eq. (20) depends on two parameters, i.e., $\alpha \geq 1$ and $\beta \geq 0$. The classical bound of the inequality is $\beta+2 \alpha$ and the maximum violation of the inequality is attained at $I_{\alpha}^{\beta}=2 \sqrt{\left(1+\alpha^{2}\right)\left(1+\frac{\beta^{2}}{4}\right)}$. For simplicity, we consider the inequality $I_{1}^{\alpha}$, known as tilted Bell-CHSH inequality and demonstrate its effect when two players share a two-qubit quantum state for playing a Bayesian game.

To the best of our knowledge, we have not encountered any Bayesian game representation of the tilted version of the Bell-CHSH operator as shown in Eq. (21). Therefore, here we make an attempt to represent the input-output relation of the tilted counterpart of the CHSH games as type of players-reward relation in Bayesian games.

$$
\begin{equation*}
I_{1}^{\alpha}=\alpha\left\langle A_{0}\right\rangle+\left\langle A_{0} B_{0}\right\rangle+\left\langle A_{0} B_{1}\right\rangle+\left\langle A_{1} B_{0}\right\rangle-\left\langle A_{1} B_{1}\right\rangle \tag{21}
\end{equation*}
$$

In order to collectively represent the operator into the settings of a Bayesian game, we need to incorporate the additional $\left\langle A_{0}\right\rangle$ term in the payoff table. Therefore, we assume that Alice and Bob play a tilted coordination game whenever Alice's input is $x_{A}=0$. However, for Alice's input $x_{A}=1$ and Bob's input $x_{B}=0$, the players play the usual un-tilted version of the coordination game. Further, when Alice's and Bob's inputs are 1 each, i.e., $x_{A}=x_{B}=1$, they play an untilted anti-coordination game. The players in these coordination and anticoordination games can have varying interests- either common interest or conflicting interest. Table 3 shows the payoff of Alice and Bob in a general Bayesian game where the payoffs depend on the type of players in a fashion similar to the winning input-output relation in a tilted CHSH game as defined by the tilted Bell-CHSH-type operator in Eq. (21). Here, the total payoff of the players is given by $T=u_{i}^{A}+u_{i}^{B}$ where $i \in\{1,2,3,4\}$ and $T^{\prime}=u_{5}^{A}+u_{5}^{B}$. The sum of payoffs which play role in the untilted version of the game is taken as $T$ and the sum of payoffs which contributes to the tilted version of the game is taken as $T^{\prime}$. Therefore, the value of $T^{\prime}$ depends on the term $\alpha$ of the tilted Bell-CHSH expression in Eq. (21). Assuming that the chances of $x_{A}\left(x_{B}\right)$ to be 0 and 1 are equiprobable, the sum of payoffs of Alice and Bob can be given as

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{T}{4}\left[\left(P_{00}^{00}+P_{00}^{11}\right)+\left(P_{01}^{00}+P_{01}^{11}\right)+\left(P_{10}^{00}+P_{10}^{11}\right)+\left(P_{11}^{01}+P_{11}^{10}\right)\right]+\frac{T^{\prime}}{2}\left[P_{0}^{0}-P_{0}^{1}\right] \tag{22}
\end{equation*}
$$

where $P_{i j}^{k l}=P\left(y_{A}=k, y_{B}=l \mid x_{A}=i, x_{B}=j\right)$ and $P_{m}^{n}=P\left(y_{A}=n \mid x_{A}=m\right)$. Thus, similar to the calculations in Eqs. (14) and (15), the sum of payoffs of the players can be re-expressed as

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{T}{2}+\frac{T}{8}\left[\left\langle A_{0} B_{0}\right\rangle+\left\langle A_{0} B_{1}\right\rangle+\left\langle A_{1} B_{0}\right\rangle-\left\langle A_{1} B_{1}\right\rangle+\frac{4 T^{\prime}}{T}\left\langle A_{0}\right\rangle\right] \tag{23}
\end{equation*}
$$

Table 8. Payoff of Alice and Bob in a general game setting where dependence of payoff on type of player commensurate with the input-output relation in a tilted CHSH game where $u_{1}^{A}, u_{1}^{B}, u_{2}^{A}$, $u_{2}^{B}, u_{3}^{A}, u_{3}^{B}, u_{4}^{A}, u_{4}^{B}, u_{5}^{A}$, and $u_{5}^{B}$ are non-zero

| Alice | Bob | $y_{B}=0$ |
| :---: | :---: | :---: |
| $y_{A}=1$ |  |  |
| $y_{A}=0$ | $u_{1}^{A}+\overline{x_{A}} u_{5}^{A}, u_{1}^{B}+\overline{x_{A}} u_{5}^{B}$ | $\overline{x_{A}} u_{5}^{A}, \overline{x_{A}} u_{5}^{B}$ |
| $y_{A}=1$ | $-\overline{x_{A}} u_{5}^{A},-\overline{x_{A}} u_{5}^{B}$ | $u_{2}^{A}-\overline{x_{A}} u_{5}^{A}, u_{2}^{B}-\overline{x_{A}} u_{5}^{B}$ |

(a) $x_{A} \cdot x_{B}=0$

| Alice | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | 0,0 | $u_{3}^{A}, u_{3}^{B}$ |
| $y_{A}=1$ | $u_{4}^{A}, u_{4}^{B}$ | 0,0 |

(b) $x_{A} \cdot x_{B}=1$
which is equal to $\frac{T}{2}+\frac{T}{8}\left(I_{1}^{\alpha}\right)$ at $\alpha=\frac{4 T^{\prime}}{T}$. In order to exemplify the payoffs of Table 8 in detail, we consider $T=2$ and thus $T^{\prime}=\frac{\alpha T}{4}=\frac{\alpha}{2}$. As described above, there can be two types of games, or simply, two ways of representation: common interest games and conflicting interest games. In a common interest game $u_{i}^{A}=u_{i}^{B}$ for $i \in\{1,2,3,4,5\}$, but in a conflicting interest game, $u_{i}^{A} \neq u_{i}^{B}$, thus creating a conflict in interest of the two players in preferring one strategy over the other.

## 7 Common interest game for tilted CHSH operator

In a common interest games, both players have the same payoff for any strategy set. As an instance, assume the utilities as $u_{1}^{A}=u_{1}^{B}=u_{2}^{A}=u_{2}^{B}=u_{3}^{A}=u_{3}^{B}=u_{4}^{A}=u_{4}^{B}=1$, and $u_{5}^{A}=u_{5}^{B}=\frac{\alpha}{4}$. The payoffs of Alice and Bob are given by Tables 9(a), 9(b), and 9(c). Clearly, Table 9(a) represents a tilted common interest coordination game and Table 9(b) represents a usual/ non-tilted common interest coordination game. Both games (Table 9(a) and (b)) have $\left\{y_{A}=0, y_{B}=0\right\}$ and $\left\{y_{A}=1, y_{B}=1\right\}$ as the two Nash Equilibriums (NEs). In case of the non-tilted version, both NEs are pareto-optimal. But, in case of tilted common interest game, one of the NEs is not pareto-optimal due to extra 'tilt' factor $\alpha$. Similarly, Table 9(c) shows a common interest anti-coordination game.

### 7.1 Classical scenario

Out of 16 different classical strategies, there can be four pareto-optimal NEs as follows

- $y_{A}=y_{B}=0$ irrespective of the values of $x_{A}$ and $x_{B}$;
- $y_{A}=0$ irrespective of the values of $x_{A}$ and $y_{B}=x_{B}$;
- $y_{A}=x_{A}$ and $y_{B}=0$ irrespective of the value of $x_{B}$; and
- $y_{A}=x_{A}$ and $y_{B}=\overline{x_{B}}$.

Each of the above NE strategy is preferred by none of the players as each gives them a payoff of $\frac{6+\alpha}{8}$. In addition, there are two non-pareto optimal NE for the common interest game, i.e.,

Table 9. Payoff of Alice and Bob in a common interest game setting where dependence of payoff on type of player commensurate with the input-output relation in a tilted CHSH game

| Alice Bob | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | $1+\frac{\alpha}{4}, 1+\frac{\alpha}{4}$ | $\frac{\alpha}{4}, \frac{\alpha}{4}$ |
| $y_{A}=1$ | $-\frac{\alpha}{4},-\frac{\alpha}{4}$ | $1-\frac{\alpha}{4}, 1-\frac{\alpha}{4}$ |

(a) $\left(x_{A}=0, x_{B}=0\right)$ or $\left(x_{A}=0, x_{B}=1\right)$

| Alice Bob | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | 1,1 | 0,0 |
| $y_{A}=1$ | 0,0 | 1,1 |

(b) $\left(x_{A}=1, x_{B}=0\right)$

| Alice Bob | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | 0,0 | 1,1 |
| $y_{A}=1$ | 1,1 | 0,0 |

(c) $\left(x_{A}=1, x_{B}=1\right)$

- $y_{A}=\overline{x_{A}}$ and $y_{B}=1$ irrespective of the value of $x_{B}$; and
- $y_{A}=y_{B}=1$ irrespective of the values of $x_{A}$ and $x_{B}$.

Both NEs are again not preferred by any player, and each yields a payoff of $\frac{6-\alpha}{8}$. For the above defined game, the maximum total payoff achieved by opting any of the Nash equilibrium strategies is $\frac{6+\alpha}{4}$. However, the average total classical NE payoff is $\frac{18+\alpha}{12}$.

The quantum strategy for this game is similar to the game where players have conflicting interests, and hence is merged with the next section.

## 8 Conflicting interest game for tilted Bell-CHSH operator

In order to better understand the representation of tilted CHSH game as a conflicting interest game, we assume the following values of utilities, $u_{1}^{A}=u_{2}^{B}=u_{3}^{B}=u_{4}^{A}=\frac{1}{2}, u_{1}^{B}=u_{2}^{A}=u_{3}^{A}=$ $u_{4}^{B}=\frac{3}{2}$, and $u_{5}^{A}=u_{5}^{B}=\frac{\alpha}{4}$. The payoffs of Alice and Bob engaging in the above defined game are given in Tables 10(a), 10(b), and 10(c). Clearly, Table 10(a) represents tilted Battle-ofSexes game, and Table 10(b) represents the usual Battle-of-Sexes game. Thus, for $x_{A} \cdot x_{B}=0$, the players engage in a $\operatorname{BoS}$ game, with an exception at $x_{A}=0$, where the players play a tilted version of the $\operatorname{BoS}$ game. Still for both versions of the game, both $\left\{y_{A}=0\right.$ (Activity 0 by Alice), $y_{B}=0$ (Activity 0 by Bob) $\}$ and $\left\{y_{A}=1\right.$ (Activity 1 by Alice), $y_{B}=1$ (Activity 1 by Bob) \} are the pareto-optimal Nash Equilibrium (NE) of the game. However, the first NE is preferred by Bob and the second NE is preferred by Alice. Also, due to the tilt $\alpha$ at $x_{A}=0$, the total payoff of the first NE (preferred by Bob) is more than the total payoff of the second NE (preferred by Alice).

At $x_{A}=x_{B}=1$, the players play an anti-coordination game which we term as a lottery game, where $y_{A}=0$ or $y_{B}=0$ corresponds to winning a bigger/ more desired prize, and $y_{A}=1$ or $y_{B}=1$ corresponds to winning a smaller/ less desired prize. Apparently, the desire of both players to win the bigger prize is the same, and both win different prizes at a time on the basis of a lottery. This leads to a conflicting-interest anti-coordination game as depicted in Table 10(c).

Table 10. Payoff of Alice and Bob in a conflicting interest game setting where dependence of payoff on type of player commensurate with the input-output relation in a tilted CHSH game

| Alice Bob | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | $\frac{1}{2}+\frac{\alpha}{4}, \frac{3}{2}+\frac{\alpha}{4}$ | $\frac{\alpha}{4}, \frac{\alpha}{4}$ |
| $y_{A}=1$ | $-\frac{\alpha}{4},-\frac{\alpha}{4}$ | $\frac{3}{2}-\frac{\alpha}{4}, \frac{1}{2}-\frac{\alpha}{4}$ |

(a) $\left(x_{A}=0, x_{B}=0\right)$ or $\left(x_{A}=0, x_{B}=1\right)$

| Alice Bob | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | $\frac{1}{2}, \frac{3}{2}$ | 0,0 |
| $y_{A}=1$ | 0,0 | $\frac{3}{2}, \frac{1}{2}$ |

(b) $\left(x_{A}=1, x_{B}=0\right)$

| Alice Bob | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | 0,0 | $\frac{3}{2}, \frac{1}{2}$ |
| $y_{A}=1$ | $\frac{1}{2}, \frac{3}{2}$ | 0,0 |

(c) $\left(x_{A}=1, x_{B}=1\right)$

### 8.1 Classical scenario

After analysing all possible classical strategy sets, one can conclude that for no classical strategy, the total payoff of the players exceeds $\frac{6+\alpha}{4}$. The strategy sets forming the Nash Equilibrium (NE), however, are different for the different type of interest (common or conflicting) of players. In case of the conflicting interest game discussed above, there are two pareto-optimal Nash equilibria, i.e.,

- $y_{A}=y_{B}=0$ irrespective of the values of $x_{A}$ and $x_{B}$; and
- $y_{A}=x_{A}$ and $y_{B}=0$ irrespective of the value of $x_{B}$.

Both strategies lead to a pareto-optimal Nash equilibrium (NE). Both these NE strategies are still preferred by Bob since Alice gets a payoff of $\frac{3+\alpha}{8}$, and Bob gets a payoff of $\frac{9+\alpha}{8}$.

Furthermore, for $\alpha \leq 1$, there are two more pareto-optimal Nash equilibria for the conflicting interest game, i.e.,

- $y_{A}=\overline{x_{A}}$ and $y_{B}=x_{B}$; and
- $y_{A}=1$ irrespective of the value of $x_{A}$, and $y_{B}=\overline{x_{B}}$.

Both strategies lead to a pareto-optimal NE strategy preferred by Alice wherein Alice gets a payoff of $\frac{7-\alpha}{8}$, and Bob gets a payoff of $\frac{5-\alpha}{8}$. Since there are two additional NE at $\alpha \leq 1$, the average total classical NE payoff is $\alpha$-independent $\left(\frac{3}{2}\right)$ for $\alpha \leq 1$ and $\alpha$-dependent $\left(\frac{6+\alpha}{4}\right)$ for $\alpha>1$.

## 9 Analysis of the tilted CHSH game for different quantum states

The games represented in Table 9 or 10 refer to Bayesian games where the total sum of payoff of both players (Alice and Bob) is equal to the value of operator $1+\frac{I_{1}^{\alpha}}{4}$ where $I_{1}^{\alpha}$ is the value of the tilted Bell-CHSH operator.

### 9.1 Quantum scenario using a pure state

When the players share a general two-qubit entangled pure state $|\psi\rangle=\cos \theta|00\rangle+\sin \theta|11\rangle$, and perform measurements as shown in Eq. (6) as strategies in the game, where $\left|\Phi_{0}(\theta)\right\rangle=$ $\cos \theta|0\rangle+\sin \theta|1\rangle$ and $\left|\Phi_{1}(\theta)\right\rangle=-\sin \theta|0\rangle+\cos \theta|1\rangle$ the total payoff of both players after their respective moves/measurements is estimated as

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{1}{4}[\alpha \cos 2 \theta+4+2 \cos 2 \lambda+2 \sin 2 \theta \sin 2 \lambda] \tag{24}
\end{equation*}
$$

Similar to the previous cases, same payoff can be achieved when players share an arbitrary twoqubit entangled state $\left|\psi_{\text {arbitrary }}\right\rangle=\cos \theta|00\rangle+e^{i \phi} \sin \theta|11\rangle$. This aggregated payoff achieves its maximum at measurement parameter $\lambda=\frac{1}{2} \tan ^{-1}(\sin 2 \theta)$, and can be given by

$$
\begin{equation*}
\$_{A}+\$_{B}=\frac{1}{4}\left[\alpha \cos 2 \theta+4+2 \sqrt{1+\sin ^{2} 2 \theta}\right] \tag{25}
\end{equation*}
$$

Optimizing with respect to $\theta$, shows that the optimal value of $\$_{A}+\$_{B}=1+\frac{1}{4} \sqrt{8+2 \alpha^{2}}$ can be obtained for a non-maximally entangled state where $\theta=\frac{1}{2} \sin ^{-1} \sqrt{\frac{4-\alpha^{2}}{4+\alpha^{2}}}$. Moreover, as the value of $\alpha$ increases from 0 to 2 , the classically attained sum of payoffs increases from 1.5 to 2. Also, with increase in $\alpha$, the angle $\theta$ at which Alice and Bob benefit with highest possible payoff decreases. At $\alpha=0$, the maximally entangled Bell state gives the maximum possible payoff to the players, as the total payoff holds correspondence with the original CHSH inequality. But at $\alpha \sim 2$, a quantum state very close to a separable state gives the highest total payoff of Alice and Bob. Apart from this, for higher $\alpha$ the maximally entangled state gives no benefit over classical strategies. However, non-maximally entangled states still give better payoff than the classically attained payoff. Therefore, our analysis suggests an interesting anomaly where players in this quantum game can achieve higher payoff by sharing a non-maximally entangled pure state instead of a maximally entangled pure state of two qubits in line with [66]. For comparison between the two game settings, one can analyse that the quantum game where conflicting interest games are merged as a Bayesian game results in a much larger set of non-maximally entangled states offering advantage over the maximally entangled state as opposed to the quantum game where common interest games are merged. Thus, this model is a clear instance where high randomness in non-maximally entangled pure states help quantum players benefit over their classical counterparts.

For further demonstration of the dependence of total payoff of the degree of entanglement, we plot the total payoff / social welfare of the players with the entanglement measure (concurrence) in Figure (3). It can be visualized that the range of total social welfare achieved using quantum strategy (sharing general 2-qubit Bell state) increases with increase in $\alpha$. Also interestingly, as the value of $\alpha$ increases, general Bell state with lesser degree of entanglement (here, concurrence) yields maximum social welfare.

### 9.2 Quantum scenario using a mixed state

In order to study the quantum scenario using mixed states, we first consider Horodecki states. Figure (4) clearly shows that as $\alpha$ increases, highest total quantum social welfare achieved increases from 1.707 to 2 . Moreover for $\alpha<1$, the highest quantum advantage is achieved

(a) $\alpha=0.25$

(c) $\alpha=0.75$

(e) $\alpha=1.25$

(g) $\alpha=1.75$

(b) $\alpha=0.5$

(d) $\alpha=1$

(f) $\alpha=1.5$

(h) $\alpha=1.99$

| -classical bound according to tilted Bell-CHSH inequality <br> ------a average classical NE payoff of game in Table IX <br> .......... average classical NE payoff of game in Table $X$ <br> ----- average classical NE payoff of game in Table XI <br> * average quantum NE payoff of game in Table IX and $X$ <br> - average quantum NE payoff of game in Table XI |
| :---: |
|  |  |

Fig. 3. Relation between the sum of payoffs of the players with the concurrence of general two-qubit Bell state for Bayesian game representation of tilted Bell-CHSH operator
through Horodecki state with maximum concurrence value ( $=1$ ), whereas for $\alpha \geq 1$, the highest social welfare is achieved through Horodecki state with minimum concurrence $(=0)$. To be precise, maximally entangled Bell state yields highest quantum welfare at $\alpha=0$, and product state yields highest quantum welfare at $\alpha=1$. This is in compliance with what we have already observed for pure states. Thus, we can say in general that with increase in $\alpha$, Horodecki states with less degree of entanglement are more useful than the sates with high entanglement. Furthermore, there is always a range of Horodecki states for which the classical welfare exceeds the quantum mechanically achieved total welfare, even though the states are entangled (and have non-zero concurrence). Interestingly, Werner states lead to similar results as in the previous section as the total payoff attained does not depend on the parameter $\alpha$.

We now analyse the total payoff in the game by sharing an efficient class of two-qubit mixed states [45], represented as

$$
\begin{equation*}
\rho=\frac{1}{N}\left[\frac{1}{2} \gamma(1-\eta)\{\gamma(1-\eta)|00\rangle\langle 00|+|01\rangle\langle 01|+|10\rangle\langle 10|\}+\left|\phi^{+}\right\rangle\left\langle\phi^{+}\right|\right] \tag{26}
\end{equation*}
$$

where $\left|\phi^{+}\right\rangle=\frac{1}{\sqrt{2}}[|00\rangle+|11\rangle], \gamma$ represents the amplitude-damping noise parameter, $\eta$ represents the weak measurement strength parameter $[68,69]$, and $N=\frac{1}{2}[2+\gamma(1-\eta)\{2+\gamma(1-\eta)\}]$ . The proposed quantum state is entangled for all values of $\gamma$ and $\eta$, but violates the BellCHSH inequality only for the range: $\max \left\{0,1-\frac{0.2428}{\gamma}\right\}<\eta<1$ [45]. In comparison to the original Bell-CHSH inequality, tilted CHSH inequality is violated for a slightly bigger range of values of $\eta$. However, Figure (5) shows that as the value of tilt increases, Alice and Bob achieve quantum advantage for a smaller range of concurrence close to unity. Even at $\alpha=0.25$, only the mixed state with $0.71<C<1$ help attain better payoff than classical strategy. Thus, a very small set of mixed states benefit the quantum players in this game setting. Nevertheless in comparison to Horodecki states, use of these states result in attaining lesser payoff in the game. For measurement strategies, we consider the orthogonal basis vectors as $\left|\Phi_{0}(\theta)\right\rangle=\cos \theta|0\rangle+e^{i \phi} \sin \theta|1\rangle$ and $\left|\Phi_{1}(\theta)\right\rangle=-e^{-i \phi} \sin \theta|0\rangle+\cos \theta|1\rangle$ at $\phi=0^{\circ}$ so as to attain maximum possible total payoff.

## 10 Conflicting interest game for tilted Bell-CHSH operator involving Chicken game

Similar to the discussion above, here we consider to study the tilted version of the game represented in Table 7. Here, the players play a tilted BoS game when the type of atleast one player is Type 0, and a Chicken game otherwise. The payoff table (Table 11) of the game is shown below.

There are three Nash equilibrium for the game, i.e.,

- $y_{A}=x_{A}$ and $y_{B}=0$ irrespective of the value of $x_{B}$ : This strategy leads to a paretooptimal Nash equilibrium preferred by Alice since Alice gets a payoff of $\frac{7+\alpha}{8}$ and Bob gets a payoff of $\frac{5+\alpha}{8}$;
- $y_{A}=y_{B}=1$ irrespective of the value of $x_{A}$ and $x_{B}$ : This strategy leads to a non


Fig. 4. Relation between the sum of payoffs of the players in tilted CHSH game with concurrence of the Horodecki state for Bayesian game representation of tilted Bell-CHSH operator (The legend is same as in Fig. 3)

Table 11. Payoff of Alice and Bob in a conflicting interest game setting involving Chicken game as an example of anti-coordination game where dependence of payoff on type of player commensurate with the input-output relation in a tilted CHSH game

| Alice | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | $\frac{1}{2}+\frac{\alpha}{4}, \frac{3}{2}+\frac{\alpha}{4}$ | $\frac{\alpha}{4}, \frac{\alpha}{4}$ |
| $y_{A}=1$ | $-\frac{\alpha}{4},-\frac{\alpha}{4}$ | $\frac{3}{2}-\frac{\alpha}{4}, \frac{1}{2}-\frac{\alpha}{4}$ |

(a) $\left(x_{A}=0, x_{B}=0\right)$ or $\left(x_{A}=0, x_{B}=1\right)$

| Alice Bob | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | $\frac{1}{2}, \frac{3}{2}$ | 0,0 |
| $y_{A}=1$ | 0,0 | $\frac{3}{2}, \frac{1}{2}$ |

(b) $\left(x_{A}=1, x_{B}=0\right)$

| Alice Bob | $y_{B}=0$ | $y_{B}=1$ |
| :---: | :---: | :---: |
| $y_{A}=0$ | 0,0 | $\frac{-1}{2}, \frac{5}{2}$ |
| $y_{A}=1$ | $\frac{5}{2}, \frac{-1}{2}$ | $-1,-1$ |

(c) $\left(x_{A}=1, x_{B}=1\right)$
pareto-optimal Nash equilibrium preferred by no player. Herein, Alice gets a payoff of $\frac{7-\alpha}{8}$ and Bob gets a payoff of $\frac{3-\alpha}{8}$;

- At $\alpha \geq 1$, the strategy $y_{A}=0$ and $y_{B}=x_{B}$ leads to a pareto-optimal Nash equilibrium preferred by Bob since Alice gets a payoff of $\frac{\alpha+1}{8}$ and Bob gets a payoff of $\frac{\alpha+11}{8}$; and
- At $\alpha \leq 1$, the strategy $y_{A}=\overline{x_{A}}$ and $y_{B}=x_{B}$ leads to a non pareto-optimal Nash equilibrium preferred by none of the players since Alice gets a payoff of $\frac{3-\alpha}{8}$ and Bob gets a payoff of $\frac{9-\alpha}{8}$

For the above defined game, the maximum total payoff achieved by opting any NE strategies is $\frac{6+\alpha}{4}$, However, the average total classical NE payoff is $\frac{17-\alpha}{12}, \frac{23}{16}$, and $\frac{17+\alpha}{12}$ for $\alpha<1, \alpha=1$, and $\alpha>1$, respectively.

The payoffs of the players playing this game is shown by dashed line in Fig. (3) for a general two-qubit Bell state; in Fig. (4) for a Horodecki state; and in Fig. (5) for the mixed state represented in Eq. (26). It is clear from these figures that the introduction of Chicken game in the Bayesian representation of tilted Bell-CHSH inequality, reduces the payoff for pure as well as mixed states. The behaviour of resources in this game is similar to the game discussed above in Tables 9 and 10. However, the dashed line in Figure (3) indicates that the set of non-maximally entangled states that perform better than the maximally entangled Bell state increases under this setting. In fact, for a fixed range of $\alpha$, i.e., $\alpha \geq 1$, all non-maximally entangled Bell-type states give higher social welfare than maximally entangled Bell state. On the other hand, Fig. (4) and (5) demonstrate that the set of mixed states which perform better than the classical strategies increases for this representation.

## 11 Bayesian games when the players share d-dimensions pure bipartite states

It has been proved that all d-dimensions pure bipartite entangled states exhibit [ $\{3, d\},\{4, d\}]$ self testing correlations [46]. Therefore, on similar lines our analysis of Bayesian games can be extended for quantum players sharing higher dimensions pure bipartite states $\sum_{i=0}^{d-1} c_{i}|i i\rangle$. A $[\{3, d\},\{4, d\}]$ Bell scenario is considered, wherein Alice and Bob have three $(x \in\{0,1,2\})$ and


Fig. 5. Relation between the sum of payoffs of the players in tilted CHSH game with concurrence of the defined mixed state for Bayesian game representation of tilted Bell-CHSH operator (The legend is same as in Fig. 3)


Fig. 6. Payoff tables for inputs $x, y$ and outputs $a, b$ of Alice and Bob respectively in 4-dimensions Bayesian game having Bell-CHSH correspondence
four $(y \in\{0,1,2,3\})$ input choices, respectively. The two players then generate outputs $a, b$ $\in\{0,1,2, \ldots, d-1\}$, which govern the strategies opted by the quantum players in the game set-up. Any two-dimensions pure bipartite state $\left(c_{i}|i i\rangle+c_{j}|j j\rangle\right)$ can be self tested using the original Bell-CHSH inequality [46] which involves two inputs (measurements) for each player (one qubit). Despite of constructing correlations for all 12 input combinations of $x$ and $y$, it is sufficient to construct correlations for $x, y \in\{0,1\}$ and $x \in\{0,2\}, y \in\{2,3\}$ for self testing a pure qudit bipartite state. Thus, the formation of a (common or conflicting interest) Bayesian game having correspondence with (tilted or untilted) d-dimensions Bell-CHSH inequality will involve eight $d \times d$ payoff tables. Figure (6) demonstrates the design of payoff tables of a Bell-CHSH Bayesian game, where Alice and Bob share a 4-dimensions pure bipartite state $|\psi\rangle_{4 d}=c_{0}|00\rangle+c_{1}|11\rangle+c_{2}|22\rangle+c_{3}|33\rangle$. Furthermore, d-dimensions Bell-CHSH expression will comprise of $\left\lceil\frac{d-3}{2}\right\rceil$ inequalities. However, a tilted version of d-dimensions Bell-CHSH expression will comprise of $2\left\lceil\frac{d-3}{2}\right\rceil$ inequalities. Additionally, the tilted expression contains $\left\lceil\frac{d-3}{2}\right\rceil$ tilt parameters in those inequalities. Similar to the games described in the preceding sections, all these inequalities can be collectively represented in a Bayesian game such that it offers quantum advantage to the players, each possessing d-dimensions particle of a pure bipartite state. One can also check the performance of the d-dimensions Bayesian game, when different mixed bipartite qudit states are shared among the players.

## 12 Conclusion and Future Scope

In last two decades, the connection between the apparently unrelated fields of Bayesian games and quantum nonlocality has been studied in great details to understand the foundations of quantum mechanics and quantum information. In general, Bayesian games in the setting of a CHSH game whose foundation makes use of quantum correlations with the purpose of defeating classical players, rely on maximally entangled states. In this work, we use general two-qubit Bell states and mixed states to analyse the game proposed by Pappa et al. It was found that all pure two-qubit entangled Bell states, set of Werner and Horodecki class states offer benefit when used as resources in comparison to classical strategies. Precisely, the players have higher advantage when they shared any general two-qubit maximally or non-maximally entangled pure Bell state over a Werner or a Horodecki class states.

Our analysis for a fully conflicting interest Bayesian game as opposed to the game designed by Pappa et al. resulted in some interesting observations. The designed game is a replica of Battle of the Sexes game when type of atleast one player is Type 0 , otherwise the game enacts a Chicken or a Hawk-Dove game. Similar to the previous case, we found that all pure states help attain higher payoff than the classical bound. Interestingly, Alice and Bob achieved higher total payoff when they shared a non-maximally entangled Bell state $\left(\cos \left(40.188^{\circ}\right)|00\rangle+\right.$ $\sin \left(40.188^{\circ}\right)|11\rangle$ ) with concurrence 0.986 rather than sharing a maximally entangled Bell state with unit concurrence. This anomaly can be attributed to an extra term in the total payoff when a Chicken game was involved, which leads to a less entangled state giving a higher payoff. Although mixed states are found to be useful as opposed to classical strategies for certain ranges of states parameters, our results suggested that quantum strategies may be more useful than classical strategies even in the range where mixed states do not violate the

Bell-CHSH inequality. This is due to the presence of non-zero entanglement in the shared state. However, mere non-zero entanglement in a mixed state may not guarantee a team of quantum players a win over their classical opponents.

Inspired by the game set-up, we formulated a general Bayesian game representation of the tilted Bell-CHSH inequality. We observed a similar phenomenon of higher payoff with less entangled states when we designed a Bayesian game based on the tilted Bell-CHSH inequality. Still, the analysis with the Bayesian game based on the standard Bell-CHSH inequality turned out to be more interesting due to the belief that maximally entangled pure Bell states are always more efficient than the non-maximally entangled pure states. Similar to the previous case, the combination of conflicting interest games led to interesting observations as opposed to the combinations of common interest games. Due to the uniqueness of tilted CHSH game, highly random non-maximally entangled states lead to higher payoff in the game, than maximally entangled Bell state. The same however is not true for all mixed states, e.g., Werner states are found to be independent of the tilt parameter and hence add no new interpretation in comparison to the Bayesian game based on the standard CHSH inequality. The Horodecki states are found to be slightly more useful resources for tilted game in comparison to set of mixed states represented in Eq. (26). As the value of tilt parameter increases, classical strategies, however, lead to better efficiency in the game against the use of these mixed states.

As an extension to our work, we have also proposed a simple layout for a two-player d-dimensions Bayesian game which can utilize the standard as well as tilted Bell-CHSH correlations present in a pure bipartite state. As a part of quantum strategy, we have specifically assumed that the players in the game perform one-parameter quantum measurements on their shared qubits. In future, one can further extend the strategy space, and generalize over more parameters to provide an extensive insight into the usefulness of different pure and mixed states in fully conflicting as well as Dilemma-involving Bayesian games. In addition, one can extensively study the usefulness and relation between randomness, nonlocal correlations, and entanglement inherent in different quantum states using the proposed Bayesian game set-up.

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